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The development of horizontal boundary layers in stratified flow. Part 1. Non-diffusive flow

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The development of the boundary layer on the upper surface of a horizontal flat plate in a non-diffusive, stratified flow is described. It is shown that the flow can be characterized by two basic parameters, the Reynolds (R_L) and Russell (Ru_L) numbers, and that, depending on the relative magnitude of these two parameters, three different régimes of flow can be defined. The delineation of these régimes and the description of the flow in each of them is obtained by deriving a uniformly valid first approximation to the Boussinesq equations of motion for a flow contained in the two-dimensional parameter space $Ru_L > 0, R_L > 1$. The critical stratification for the self-blocking of a horizontal boundary layer is shown to be given by the condition $Ru_L = O(R_L^{\frac{1}{4}})$.

1. Introduction

Stratified flows in a gravitational field exhibit many remarkable phenomena which are nonexistent in the flow of homogeneous fluids. The development of the boundary layer on a horizontal plate is one example. When the stratification is large and the motion of the fluid is slow, a boundary layer whose thickness decreases in the downstream direction appears and a viscous wake exists upstream of the plate. This is in striking contrast to the familiar downstream growing boundary layer and downstream viscous wake existing when the fluid is homogeneous.

Long (1959) first observed experimentally and described theoretically the existence of a viscous wake with a multiple jet-like structure upstream of a body moving horizontally in a stratified fluid. He derived a similarity solution which is valid far upstream of an obstacle and showed that velocity perturbations relative to the horizontal free stream decay algebraically ($x^{-\frac{2}{3}}$) with distance measured upstream from the obstacle. The solution characterizes the blocking of the flow ahead of a body.

Martin (1966) and Martin & Long (1968) subsequently investigated the boundary layer above a slowly moving horizontal plate under conditions for which an upstream wake occurred. Their experiments, as well as those performed by Pao

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(1968), showed the remarkable result that the boundary-layer thickness decreased in the downstream direction. They were able to describe this flow structure theoretically by solving the equations of motion in which the advective terms and density diffusion were neglected. They also demonstrated that, when density diffusion is allowed, the diffusion boundary layer continues to grow in the downstream direction just as in the case of homogeneous flows.

The purpose of this investigation is to provide a parametric study of the influence of density stratification on the development and structure of horizontal boundary-layer regions. The appearance of upstream wakes and upstream growing boundary layers implies that a critical stratification exists for which the thickness of a downstream growing boundary layer becomes sufficiently large to induce blocking. Blocking of a flow ahead of an obstacle can be understood on the basis of energy considerations, but the occurrence of a self-blocking due solely to the action of viscosity is more difficult to understand. The establishment of a criterion for determining which boundary-layer structure appears for specified flow conditions is one of the objectives of this study.

Another interesting aspect of boundary layers in stratified media concerns the coupling between the viscous boundary layer and the outer inviscid flow. From existing studies of boundary layers in homogeneous flows, we know that the boundary layer displaces the outer flow in a direction transverse to the external flow. Since stratification effectively inhibits vertical motions, the question arises as to the interaction between the outer stratified flow and a horizontal boundary layer. Furthermore, since any non-trivial stratified flow is rotational, the boundary-layer induced perturbation on the external flow establishes a possible vorticity interaction with the boundary layer. These effects are investigated for the flow over a horizontal plate by deriving a uniformly valid solution to first order, with the magnitude of the density stratification $|d\ln\rho/dx_3|$ appearing as a parameter.

2. Formulation

We consider the development of a viscous boundary layer on the upper surface of a horizontal flat plate of length L in a stably stratified flow (as shown in figure 1). Taking the viscosity μ_0, the specific heat c_{p_0}, and the thermal conductivity k_0 to be constant, the dimensionless equations of motion for steady, low speed ($M^2 \ll 1$; M = Mach number), thermally stratified flow are

$$\nabla\cdot(\rho\mathbf{q}) = 0, \tag{1}$$

$$\rho(\mathbf{q}\cdot\nabla)\mathbf{q} = -\nabla p - \frac{\rho}{F_L}\hat{\mathbf{k}} - \frac{1}{R_L}\nabla\times(\nabla\times\mathbf{q}), \tag{2}$$

$$\rho(\mathbf{q}\cdot\nabla)T = \frac{1}{P_0 R_L}\nabla^2 T, \tag{3}$$

and

$$\rho = \rho(T). \tag{4}$$

The equations have been made dimensionless by scaling the independent variables with the plate length L, the velocity with its free-stream value U_0, the density and temperature by their respective values at the level of the plate (ρ_0 and T_0), and the pressure by the dynamic head ($\rho_0 U_0^2$). The three dimensionless parameters appearing in the above equations, the Froude number F_L, the Prandtl number P_0, and the Reynolds number R_L are defined as

$$F_L = \frac{U_0^2}{gL}, \quad P_0 = \frac{\mu_0 c_{p_0}}{k_0} \quad \text{and} \quad R_L = \frac{\rho_0 U_0 L}{\mu_0}. \tag{5}$$

The equation of state (4) denotes that the fluid is incompressible in that changes in pressure induce negligible changes in density. This is consistent with the restriction of the analysis to low speed ($M^2 \ll 1$) flows.

The above equations are written explicitly for thermally stratified flows, but they also describe molecularly stratified flows if T is replaced by the mass fraction of the biasing species and the Schmidt number is substituted for the Prandtl number.

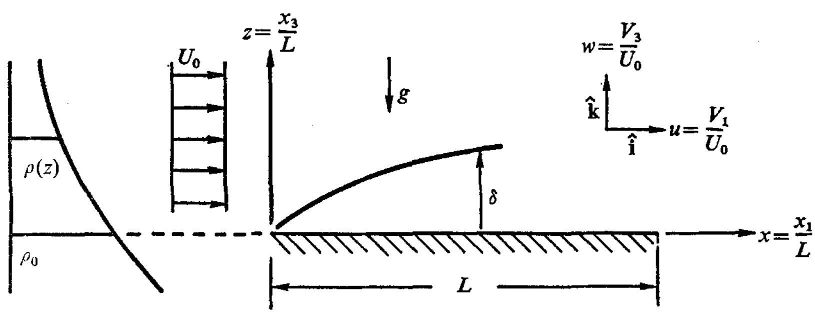

FIGURE 1. A schematic of the flow model.

The structure of the velocity field above the plate is studied first for the limiting case of a large Prandtl number. In this limit ($P_0 \to \infty$), the diffusion of heat can be neglected and the energy equation reduces to

$$(\mathbf{q}\cdot\nabla)T = 0, \tag{6}$$

or by use of the equation of state (4),

$$(\mathbf{q}\cdot\nabla)\rho = 0. \tag{7}$$

The diffusive case (arbitrary Prandtl number) is studied in part 2 of this analysis.

Combining (1) and (7), the continuity equation reduces to the incompressible form

$$\nabla\cdot\mathbf{q} = 0. \tag{8}$$

Assuming the plate is infinitely wide so that the flow can be taken as two-dimensional, (8) permits the introduction of a stream function ψ defined by

$$\mathbf{q} = -(\nabla\times\hat{\mathbf{j}})\,\psi(x,z) = \hat{\mathbf{i}}\frac{\partial\psi}{\partial z} - \hat{\mathbf{k}}\frac{\partial\psi}{\partial x}. \tag{9}$$

Equations (6) and (7) can then be integrated to yield

$$T = T(\psi) \quad \text{and} \quad \rho = \rho(\psi). \tag{10}$$

The analytic forms of $\rho(\psi)$ and $T(\psi)$ are determined by the boundary conditions far upstream of the plate. Results (9) and (10) provide a great simplification and permit the system of equations (2), (7) and (8) to be written in terms of a single equation for the stream function ψ.

Considering a density stratification given by

$$\rho(x \to -\infty, z) = \rho_s(z) = e^{-\beta z} = e^{-(\beta_0 L)x_3/L}, \tag{11}$$

and invoking the Boussinesq approximation, the vorticity equation is obtained in the form

$$\left[L(x, z, \psi) - \frac{1}{R_L}\nabla^2\right]\nabla^2\psi + Ru_L^2\psi_x = 0, \tag{12}$$

where

$$L(x, z, \psi) = \psi_z \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial z}. \tag{13}$$

The operator $L(x, z, \psi)$ appears extensively throughout the succeeding analysis and, for convenience, is written in shorthand form where the symbols in the parenthesis indicate the horizontal and vertical co-ordinate variables and the dependent variable of the operator in that order. The parameter Ru_L represents the Russell number, a designation originally ascribed by Miles (1968). It is defined as

$$Ru_L^2 = \left(\frac{NL}{U_0}\right)^2 = \frac{\beta}{F_L}, \tag{14}$$

where

$$N = \left(-g\frac{d\ln\rho_s}{dx_3}\right)^{\frac{1}{2}} = (g\beta_0)^{\frac{1}{2}}, \tag{15}$$

N denoting the intrinsic frequency. Two independent parameters appear in (12); the first parameter, R_L^{-1}, scales the viscous terms relative to the inertia terms, and the second, Ru_L^2, scales the buoyancy term relative to the inertia terms. Their relative magnitudes can be expected to play an important role in determining the flow structure in the vicinity of the plate.

The boundary conditions applicable to (12) for the problem depicted in figure 1 are

$$\begin{gathered} \psi(x, 0) = 0, \quad (x < 0), \\ \psi(x, 0) = \psi_z(x, 0) = 0, \quad (0 \leqslant x \leqslant L), \end{gathered} \tag{16}$$

and

$$\psi_z(x \to -\infty, z) = \psi_z(x, z \to \infty) = 1.$$

We now seek a uniformly valid first approximation to the solution of (12), subject to the conditions (16), for large Reynolds numbers but with the Russell number varying from small to large values.

3. The boundary-layer approximation

Consider first the flow region in the immediate vicinity of the plate where viscosity has a first-order effect. Anticipating that the vertical scale of this region is small relative to the horizontal length of the plate, we introduce the boundary-layer transformation

$$y = z/\epsilon, \quad \epsilon = \epsilon(R_L, Ru_L) \simeq \delta/L \ll 1,$$

and

$$\psi(x, z) = \epsilon\Psi(x, y), \tag{17}$$

where ϵ is a function of the parameters that appear in the differential equation (12). The functional form of ϵ is determined by requiring the coefficient of the highest order viscous term to be unity and all remaining terms in the vorticity equation to be of order unity or smaller.

Introducing the transformation (17) into the vorticity equation (12) yields the boundary-layer equation

$$\left[L(x,y,\Psi)-\frac{\epsilon^{-2}}{R_L}\left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]\left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi+\epsilon^2 Ru_L^2\Psi_x=0. \tag{18}$$

Buoyancy contributes to the vorticity balance in the boundary layer in proportion to the square of the Russell number based on the boundary-layer thickness δ, since

$$\epsilon^2 Ru_L^2=\frac{\delta^2}{L^2}\frac{N^2L^2}{U_0^2}=Ru_\delta^2. \tag{19}$$

The Russell number based on a representative vertical dimension of an obstacle characterizes the structure of the flow over the obstacle (cf. Long 1953, 1954, 1955, 1959; Miles 1968) and, when that Russell number becomes large, the flow is blocked upstream of the obstacle.

Two limiting cases of (18) are now considered. First, when the Russell number is small (small stratification, $R_L > Ru_L^2$), the boundary layer is characterized by a balance between the inertia and viscous terms with the familiar scale

$$\epsilon=\epsilon_{iv}=R_L^{-\frac{1}{2}}. \tag{20}$$

The buoyancy term is then of order (Ru_L^2/R_L). Writing the stream function as a perturbation sequence in ϵ,

$$\Psi(x,y;\epsilon)=\Psi^{(1)}(x,y)+\alpha(\epsilon)\Psi^{(2)}(x,y)+\ldots, \tag{21}$$

and substituting into (18), we obtain for $\Psi^{(1)}$ the equation

$$\left[L(x,y,\Psi^{(1)})-\frac{\partial^2}{\partial y^2}\right]\frac{\partial^2\Psi^{(1)}}{\partial y^2}=0. \tag{22}$$

This equation can be integrated once with respect to y to yield the Blasius equation (cf. Rosenhead 1963, p. 222). The solution, Ψ_B, say, obtained by means of the similarity transformation

$$\eta=\frac{y}{x^{\frac{1}{2}}},$$

$$\Psi^{(1)}(x,y)=\Psi_B=x^{\frac{1}{2}}f_1(\eta), \tag{23}$$

is well-known. A property of Ψ_B which has important consequences in the subsequent development is that the solution is not uniformly valid since

$$\left.\begin{aligned}\lim_{y\to\infty} w(x,y)&=0\cdot865\,\epsilon_{iv}x^{-\frac{1}{2}},\\ \text{or}\qquad \Psi^{(1)}(x,\infty)&=y-1\cdot730x^{\frac{1}{2}}.\end{aligned}\right\} \tag{24}$$

The second case we consider is the limit of large Russell numbers (large stratification, $R_L < Ru_L^2$). The boundary-layer scaling is then given by

$$\epsilon=\epsilon_{bv}=(R_L Ru_L^2)^{-\frac{1}{4}}. \tag{25}$$

Since inertia terms are then of order $(R_L^{\frac{1}{2}}/Ru_L)$, the first-order boundary-layer vorticity equation becomes

$$\Psi_x^{(1)} - \Psi_{yyyy}^{(1)} = 0, \tag{26}$$

which corresponds to a balance between the diffusion of vorticity and the baroclinic generation of vorticity. Equation (26) was first derived by Long (1959) in his analysis of a viscous wake upstream of an obstacle. Later, a similarity solution to (26) was obtained by Martin & Long (1968) describing the boundary layer on a horizontal flat plate. They showed that, in order to obtain a physically meaningful solution to the parabolic equation (26), the direction of the time-like variable x had to be reversed, leading to a boundary layer with upstream growth and an upstream wake. If we let

$$\bar{x} = 1 - x,$$

so that

$$\Psi_{\bar{x}}^{(1)} + \Psi_{yyyy}^{(1)} = 0,$$

the similarity solution is of the form

$$\eta = y/\bar{x}^{\frac{1}{4}},$$

$$\Psi^{(1)}(\bar{x}, y) = \Psi_L = \bar{x}^{\frac{1}{4}} f_1(\eta). \tag{27}$$

Their solution, Ψ_L, is uniformly valid in that the vertical velocity approaches zero exponentially fast at the outer edge of the boundary layer.

A useful representation of the above results which clarifies the interplay between the two parameters Ru_L and R_L is obtained by replacing the Russell number by a power of the Reynolds number,

$$Ru_L^2 = R_L^n. \tag{28}$$

The vertical scale ϵ of the first-order boundary layers for the inertia-viscous balance (22) and the buoyancy-viscous balance (27) are then given by

$$\epsilon_{iv} = R_L^{-\frac{1}{2}},$$

$$\epsilon_{bv} = R_L^{-\frac{1}{4}(n+1)},$$

so that

$$\frac{\epsilon_{bv}}{\epsilon_{iv}} = R_L^{\frac{1}{4}(1-n)}. \tag{29}$$

Using the latter relation, we can delineate three distinct boundary-layer types depending on the relative magnitude of the Russell and Reynolds numbers. For $n < 1$, $\epsilon_{iv} < \epsilon_{bv}$ and the first-order boundary layer is the Blasius one (22), in which convection and diffusion of vorticity are balanced. When $n > 1$, $\epsilon_{bv} < \epsilon_{iv}$, and the first-order boundary layer is described by Long's equation (27). The third boundary-layer type occurs when the condition $n = 1$ is satisfied. In this case $\epsilon_{iv} = \epsilon_{bv}$ and convection, diffusion, and baroclinic generation of vorticity are all of equal order in the boundary layer. The governing first-order equation then becomes

$$\left[L(x, y, \Psi^{(1)}) - \frac{\partial^2}{\partial y^2}\right]\frac{\partial^2 \Psi^{(1)}}{\partial y^2} + \Psi_x^{(1)} = 0. \tag{30}$$

We refer to this case as the critical boundary layer since it is transitional between a downstream growing boundary layer ($n < 1$) and an upstream growing boundary layer ($n > 1$). A similarity solution of (30) is possible only for the case of an accelerated flow ($U_e \sim x^{\frac{1}{2}}$).

The above three equations, (22), (27), and (30) describe all the possible first-order boundary layers on a horizontal surface in a stratified flow. Their classification depends strongly on the relative magnitude of the Russell and Reynolds numbers. To obtain a uniformly valid approximation to the entire flow structure, however, we must also examine the outer flow, to which the solutions of the above equations must match.

4. The outer flow

In considering the outer flow, we use the Russell–Reynolds number relation (28) and equation (12). The stream function expansion for the outer flow is

$$\psi(x, z; \epsilon) = z + \delta_1(\epsilon)\, \psi^{(1)}(x, z) + \ldots, \tag{31}$$

where the first term on the right describes the zeroth-order motion. The gauge function $\delta_1(\epsilon)$ is equal to ϵ and takes on the value dictated by the first-order boundary layer as given in (29). Substituting the above expansion into (12) yields the following equation for the outer flow

$$\left[\frac{\partial}{\partial x} - \frac{1}{R_L}\nabla^2\right] \nabla^2 \psi^{(1)} + R_L^n \psi_x^{(1)} = O(\epsilon). \tag{32}$$

Examining this equation, we again find that there are three different cases depending on the value of the exponent n, i.e. on the magnitude of the Russell number. When $n < 0$, the last term on the left-hand side is smaller than unity and, in fact, vanishes in the limit ($R_L \to \infty$). The first-order outer flow is then governed by the equation

$$\frac{\partial}{\partial x} \nabla^2 \psi^{(1)} = 0 \quad (n < 0). \tag{33}$$

The outer flow in this case is determined by a balance between the inertia and pressure forces while the buoyancy and viscous terms appear only in higher order equations. When $n = 0$, the inertia and buoyancy terms are of equal importance, and the first-order outer flow is described by the equation

$$\frac{\partial}{\partial x}[\nabla^2 + 1]\psi^{(1)} = 0 \quad (n = 0). \tag{34}$$

The stratification is now sufficiently large that the boundary-layer displacement effect renders the baroclinic generation of vorticity a first-order role in the outer flow. Thirdly, when $n > 0$, the buoyancy term in equation (32) dominates, and the first-order flow is governed by the equation

$$\frac{\partial \psi^{(1)}}{\partial x} = 0 \quad (n > 0). \tag{35}$$

This relation is analogous to the Taylor–Proudman theorem in rotating flows and expresses the fact that the constraining influence of stratification is sufficiently large to inhibit vertical motions. The outer flow is then in hydrostatic balance regardless of the boundary-layer displacement effect.

An incompatability in the above set of equations is immediately apparent. For $n < 1$, the first-order boundary layer is described by the Blasius equation which requires that the first-order outer flow satisfy the matching condition

$$\psi^{(1)}(x,0) = -1{\cdot}730x^{\frac{1}{2}} \quad (n < 1). \tag{36}$$

However, for $n > 0$ the outer flow is governed by (35), which clearly does not admit a solution satisfying condition (36). Hence, we must conclude that for $0 < n < 1$ either the steady flow breaks down into some unsteady structure or a more complicated coupling exists involving an intermediate layer through which the Blasius solution and the solution $\psi^{(1)} = 0$ of (35) can be properly matched. We assume the latter to be true and re-examine (12) and (18) in the parameter range $0 < n < 1$. For $n > 1$, no difficulty occurs since the boundary-layer solution ψ_L from (27) is uniformly valid.

5. The intermediate layer

Examining the boundary-layer equation (22) and the outer-flow equation (35), it is clear that the outer flow is governed by a pressure-buoyancy (hydrostatic) balance, while the boundary layer is characterized by a balance between the inertia, pressure, and viscous stress terms. The importance of the buoyancy term must diminish as one approaches the plate from the free stream, and the importance of the inertia terms must diminish as one proceeds away from the plate toward the free stream. Intuitively then, one expects that a region exists between the boundary layer and the external flow wherein an inertia-pressure-buoyancy balance occurs.

To derive the correct first-order approximation to the flow in the intermediate region, we introduce the transformation

$$\hat{y} = \sigma(R_L, n)y = \frac{z}{\epsilon/\sigma}, \quad \epsilon = R_L^{-\frac{1}{2}},$$

and

$$\psi(x,z) = (\epsilon/\sigma)\hat{\Psi}(x,\hat{y}). \tag{37}$$

Substituting (37) into (12) we obtain the equation

$$\left[L(x,\hat{y},\hat{\Psi}) - \sigma^2\left\{(\epsilon/\sigma)^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \hat{y}^2}\right\}\right]\left\{\left(\frac{\epsilon}{\sigma}\right)^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \hat{y}^2}\right\}\hat{\Psi} + \sigma^2 R_L^{n-1}\hat{\Psi}_x = 0. \tag{38}$$

Choosing σ so that a proper balance of terms is maintained leads to the condition

$$\sigma = R_L^{-\frac{1}{2}(1-n)}, \tag{39}$$

whereby the inertia and buoyancy terms are balanced and the viscous stress terms are at most of order σ^2. The characteristic vertical scale Δ_I of the intermediate region is

$$\Delta_I/L = (\epsilon/\sigma) = R_L^{-\frac{1}{2}n} \quad (0 \leqslant n < 1). \tag{40}$$

When $n = 0$, the intermediate layer contains the entire outer flow and, as the stratification is increased (increasing n), the vertical extent of the layer decreases until it is completely contained within the primary boundary layer when $n = 1$.

Thus, an intermediate region described by the scaling (40) and equation (38) can be defined. However, to conclusively demonstrate the existence of a double structure for $0 < n < 1$, it is necessary to show that solutions of (38) are possible satisfying the conditions

$$\hat{\Psi}^{(1)}(x, 0) = -1{\cdot}730x^{\frac{1}{2}} \quad \text{and} \quad \hat{\Psi}^{(1)}(x, \infty) = 0. \tag{41}$$

This topic has been considered in detail by Redekopp (1969), who demonstrated that such solutions are impossible unless the horizontal co-ordinate is scaled along with the vertical co-ordinate. The x scaling that is required is exactly equivalent to the z scaling, i.e.

$$\hat{x} = \frac{x}{\epsilon/\sigma} = xR_L^{\frac{1}{2}n} \quad (0 \leqslant n \leqslant 1). \tag{42}$$

This is the only scaling which allows a consistent matching between the boundary layer and the outer flow. Observe that when $n = 1$, $\hat{x}$ is of the same order as the boundary-layer thickness, which suggests that perhaps the complete Navier–Stokes equations are required to describe the $n = 1$ case. This presents a plausible explanation as to how the transition between the two parabolic cases (22) for $(n < 1)$ and (29) for $(n > 1)$ is accomplished.

A justification for the scaling (42) is provided by the following consideration. Outside the primary boundary layer, the representative length for the flow is no longer that of the body (L), but the characteristic wavelength of internal waves. This is precisely what the scaling (42) accomplishes, as can be seen by defining a length λ equivalent to the length of a wave oscillating at the intrinsic frequency N and moving with velocity U_0,

$$\lambda = \frac{U_0}{N} = \frac{L}{Ru} = LR_L^{-\frac{1}{2}n}.$$

Rescaling the x variable with λ we obtain

$$\hat{x} = x\bigg/\left(\frac{\lambda}{L}\right) = xR_L^{\frac{1}{2}n}. \tag{43}$$

The order of magnitude of the viscous terms is then

$$\epsilon\sigma = R_L^{-1+\frac{1}{2}n} \tag{44}$$

which is of order (R_L^{-1}) as in the case of homogeneous flow when $n = 0$ and of order $(R_L^{-\frac{1}{2}})$ when $n = 1$.

The stream function expansion for the intermediate layer is of the form

$$\psi(x, z; R_L) = \frac{\epsilon}{\sigma}\hat{\Psi}(\hat{x}, \hat{y}) = R_L^{-\frac{1}{2}n}[\hat{y} + \gamma(R_L)\hat{\Psi}^{(1)}(\hat{x}, \hat{y}) + \ldots], \tag{45}$$

where the form of $\gamma(R_L)$ is chosen so that $\hat{\Psi}^{(1)}(\hat{x}, \hat{y})$ matches to the Blasius solution Ψ_B. Carrying through the matching yields

$$\gamma(R_L) = \epsilon\left(\frac{\epsilon}{\sigma}\right)^{-\frac{1}{2}} = R_L^{-\frac{1}{2}+\frac{1}{4}n}$$

and

$$\hat{\Psi}^{(1)}(\hat{x}, \hat{y} = 0) = -1{\cdot}730\hat{x}^{\frac{1}{2}}. \tag{46}$$

The first-order equation for the intermediate layer then becomes

$$\frac{\partial}{\partial \hat{x}}[\hat{\nabla}^2+1]\hat{\Psi}^{(1)} = 0. \tag{47}$$

This equation is applicable for the parameter range $0 < n < 1$. Its form, together with the boundary conditions, is identical to the first-order equation for $n = 0$.

6. The flow due to boundary-layer displacement

In this section we present the solution for the first-order outer flow induced by the displacement effect of the Blasius boundary layer. Since the exact shape of the displacement body in the downstream wake is unknown, we calculate the outer flow as if the plate were semi-infinite.

For Russell numbers less than unity ($n < 0$), the outer flow is potential (equation (33)), and the solution satisfying the matching condition (36) is given by Van Dyke (1964, p. 134). It can be written in the form

$$\psi^{(1)}(x,z) = -0{\cdot}865\,((x+iz)^{\frac{1}{2}}+(x-iz)^{\frac{1}{2}}) = -1{\cdot}730 r^{\frac{1}{2}}\cos\tfrac{1}{2}\theta. \tag{48}$$

For $0 \leqslant n < 1$, the outer flow is described by the Helmholtz equation (equations (34) and (47)) which we write in the form

$$\nabla^2\phi + a^2\phi = 0, \tag{49}$$

with the boundary conditions

$$\phi = o(z) \quad \text{as} \quad (x^2+z^2)\to\infty,$$

$$\phi(x,0) = 0 \quad \text{for} \quad x<0,$$

and

$$\phi(x,0) = -1{\cdot}730 x^{\frac{1}{2}} \quad \text{for} \quad x>0. \tag{50}$$

It is understood that $(\hat{x}, \hat{y})$ are substituted for (x,z) when $0 < n < 1$ and that ϕ denotes either $\psi^{(1)}$ or $\hat{\Psi}^{(1)}$ depending on the value of n. The parameter a is included to indicate explicitly the role of the Russell number.

The solution of the Helmholtz equation describing the flow of a stratified fluid over various shaped obstacles has been the concern of a number of investigators, particularly as it relates to the phenomena of internal waves in the lee of mountain ranges. Queney *et al.* (1960) and Miles (1968) have given comprehensive reviews of the existing solutions. For the solution of the boundary-value problem (49) and (50), we follow the development by Graham (1966) for the flow over an arbitrarily shaped slender body. Graham's solution is given in the form

$$\phi(x,z) = \int_0^\infty f(\xi)\,\phi_D(x-\xi,z)\,d\xi, \tag{51}$$

where $f(x)$ is the dipole density and $\phi_D(x,z)$ denotes the solution of (49) for an isolated dipole of strength b

$$\phi_D(x,z) = bz\frac{Y_1(a[z^2+x^2]^{\frac{1}{2}})}{[z^2+x^2]^{\frac{1}{2}}} + \frac{b}{\pi}\sum_{m=1}^{\infty}\frac{8m}{4m^2-1}J_{2m}(a[z^2+x^2]^{\frac{1}{2}})\sin\left(2m\tan^{-1}\frac{z}{x}\right). \tag{52}$$

In the framework of small disturbance theory, which is clearly applicable, the dipole density is related to the vertical perturbation velocity $w^{(1)}$ at the altitude $z = 0$ by

$$f(x,0) = \int_0^x \tfrac{1}{2}a\, w^{(1)}(\xi,0)\, d\xi = \tfrac{1}{2}a(1\cdot730x^{\frac{1}{2}}). \tag{53}$$

Note that the dipole strength depends directly on a, or, using the basic parameters of the problem, the Russell number Ru_L. This reflects the fact that the scale of the flow outside the boundary layer must change as the Russell number increases. Since the boundary-layer displacement is independent of the stratification to this order, (53) requires that the independent variable x be scaled in such a way that the dipole strength is always comparable to the magnitude of the boundary-layer displacement velocity $w^{(1)}(x,0)$, even when the Russell number is large $(0 < n < 1)$. This is precisely what the intermediate layer scaling (43) accomplishes.

Substituting the results (52) and (53) into (51), the solution for $\phi(x,z)$ becomes

$$\phi(x,z) = -1\cdot730r^{\frac{1}{2}}\cos\tfrac{1}{2}\theta + \frac{1\cdot730a}{\pi}\sum_{m=1}^{\infty}\frac{4m}{4m^2-1}\int_0^{\infty} J_{2m}(a[z^2+(x-\xi)^2]^{\frac{1}{2}}) \times \sin\left(2m\tan^{-1}\frac{z}{x-\xi}\right)d\xi. \tag{54}$$

The first term, which is identical to the solution (48) for potential flow, derives from the integration of the first term in (52). The effect of density stratification is then contained solely in the integral term of (54).

The integral and sum in (54) were evaluated numerically by integrating between the limits $\xi = 0$ to $\xi = 100$ and taking ten terms of the sum. An upper limit of ten for the summation was chosen because it corresponds to approximately a ten-fold decrease in magnitude between the first and tenth terms. Since there is no characteristic geometrical length for a semi-infinite plate, all lengths are scaled by the stratification length

$$L = \beta_0^{-1} = \left|\frac{d\ln\rho_s}{dx_3}\right|^{-1}. \tag{55}$$

Numerical values were computed for x ranging between $x = -5$ and $x = 20$ in increments of $\Delta x = 0\cdot2$ with z ranging between $z = 0\cdot25$ and $z = 3\cdot0$. The first-order, uniformly valid solutions for $a = 1\cdot0$ and Reynolds numbers of 100 and 1000 are shown in figure 2, where

$$\psi(x,z) = z + \epsilon\psi^{(1)}(x,z) + \epsilon(y - \Psi^{(1)}(x,y)),$$

and

$$\epsilon = R_{\beta_0}^{-\frac{1}{2}} = \left(\frac{\nu_0\beta_0}{U_0}\right)^{\frac{1}{2}}. \tag{56}$$

No wave pattern appears and the streamlines exhibit the same general shape that exists for homogeneous flow. This is somewhat surprising in light of Lyra's (1943) solution for the flow over a semi-infinite plateau (forward facing step) which shows a very distinct pattern of waves. A possible explanation for this is that there is a critical bluntness for a monotonic, semi-infinite body which must be exceeded if waves are to be generated. For the flow over a finite flat plate, the thickness of the displacement body decreases downstream of the trailing edge of the plate and it is quite likely that a wave pattern would appear in the downstream flow field.

Another interesting feature of the solution (54) is that the horizontal perturbation velocity vanishes as z tends to zero indicating that there is no coupling between the first-order outer flow and the second-order boundary layer. It is worth noting that a coupling between the outer flow and the second-order boundary layer does exist when the non-Boussinesq terms are included in the outer flow equations. As will be demonstrated in part 2 of this analysis, a coupling enters via the thermal field when diffusion is allowed, even in the Boussinesq approximation.

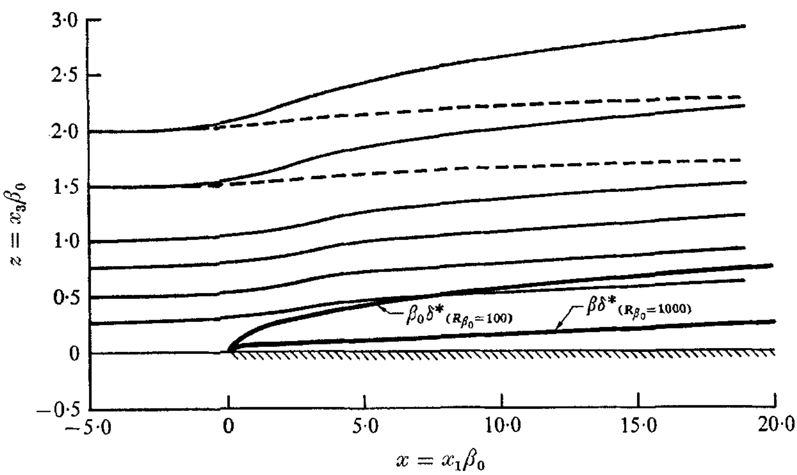

FIGURE 2. The first-order streamline pattern for the case $n = 0$. —, streamlines for $R_{\beta_0} = 100$; - - -, streamlines for $R_{\beta_0} = 1000$.

7. The second-order boundary layer

In the parameter range $0 < n < 1$, the boundary-layer expansion is given by

$$\psi(x,z) = R_L^{-\frac{1}{2}}[\Psi^{(1)}(x,y) + \alpha(R_L)\Psi^{(2)}(x,y) + \ldots], \tag{57}$$

and, from (45) and (46), the corresponding outer-flow expansion is

$$\psi(x,z) = R_L^{-\frac{1}{2}n}[\hat{y} + R_L^{-\frac{1}{2}+\frac{1}{4}n}\hat{\Psi}^{(1)}(x,y) + \ldots]. \tag{58}$$

Substituting (57) into the boundary-layer form of the vorticity equation (18), we obtain the equation

$$\left[L(x,y,\Psi^{(1)}) - \frac{\partial^2}{\partial y^2}\right]\frac{\partial^2\Psi^{(2)}}{\partial y^2} + L(x,y,\Psi^{(2)})\frac{\partial^2\Psi^{(1)}}{\partial y^2} + \frac{R_L^{n-1}}{\alpha(R_L)}\Psi_x^{(1)} = 0. \tag{59}$$

If there is a forcing of the second-order boundary layer arising from the first-order outer flow ($\hat{\Psi}^{(1)}$), the gauge function α is given by

$$\alpha = R_L^{-\frac{1}{4}n}. \tag{60a}$$

On the other hand, the forcing arising from the baroclinic term requires that

$$\alpha = R_L^{n-1}. \tag{60b}$$

The contributions from these two forcing effects are equal when $n = \frac{4}{5}$. When $n < \frac{4}{5}$ the baroclinic generation term has no influence on the solution to first order. However, when $n > \frac{4}{5}$ the second-order boundary-layer contribution due to the baroclinic generation of vorticity is more significant than the correction to the outer flow due to the displacement effect of the first-order (Blasius) boundary layer. A uniformly valid solution to first order, then, requires that $\Psi'^{(2)}(x, y)$ be evaluated for $\frac{4}{5} \leqslant n < 1$ using (59) with homogeneous boundary conditions and α given by (60*b*). For the flow over a flat plate, $\Psi'^{(2)}$ has the same form throughout the range $0 < n < 1$ since the displacement induced horizontal velocity vanishes at the plate surface (at least in the Boussinesq approximation) so that the boundary conditions and the differential equation are the same for the entire range.

Using the Blasius solution (equation (23)), a similarity solution of (59) is possible and has the form

$$\Psi'^{(2)}(x, y) = x^{\frac{3}{2}} f_2(\eta), \tag{61}$$

where $f_2(\eta)$ satisfies the equation

$$f_2^{\text{iv}} + \tfrac{1}{2} f_1 f_2''' - \tfrac{1}{2} f_1' f_2'' + \tfrac{1}{2} f_1'' f_2' + \tfrac{3}{2} f_1''' f_2 = \tfrac{1}{2}(f_1 - \eta f_1'). \tag{62}$$

The forcing term on the right-hand side is known from the Blasius solution and corresponds to the streamwise derivative of the temperature as expressed by (10). It is equal to the negative of the first-order vertical velocity and, therefore, approaches a constant value as η becomes large. Consequently, (62) reveals that the second-order shear approaches a constant for large η

$$\lim_{\eta \to \infty} f_2'' = - \lim_{\eta \to \infty} (f_1 - \eta f_1') = 1{\cdot}730. \tag{63}$$

This violates the definition of a boundary layer and indicates that another intermediate layer must exist in which the shear decays to zero. It appears that the same difficulty is encountered in higher-order terms for $n \leqslant 0$ as well. We are investigating this problem further in an attempt to resolve the difficulty (solutions for the second-order boundary layer for Prandtl number of order unity are given in part 2).

8. Summary

We have found that two characteristic parameters describe the boundary-layer flow of a stratified fluid, the Reynolds and Russell numbers, and that their relative magnitude define three different régimes of flow. These régimes are given by (i) $Ru_L < O(1)$, (ii) $O(1) \leqslant Ru_L < O(R_L^{\frac{1}{2}})$, and (iii) $Ru_L > O(R_L^{\frac{1}{2}})$. The ranges of applicability of each of these régimes are shown schematically in figure 3. In the first case the inner flow is the familiar Blasius boundary layer and the outer flow is potential. In the second case, the primary boundary-layer flow is still described by the Blasius equation, but an intermediate region exists in which the flow induced by the displacement effect of the boundary layer adjusts to a parallel outer flow. Both dependent variables must be scaled with the wavelength of waves oscillating at the natural frequency and moving with the free stream velocity in order to obtain a consistent representation of the outer flow in this régime. In the third case, the boundary layer changes from one with downstream growth to one with upstream growth. The upstream flow is then changed and, in order to maintain a balance between the diffusion of vorticity and the baroclinic generation of vorticity, the streamlines must diverge in the downstream direction and an upstream wake appears.

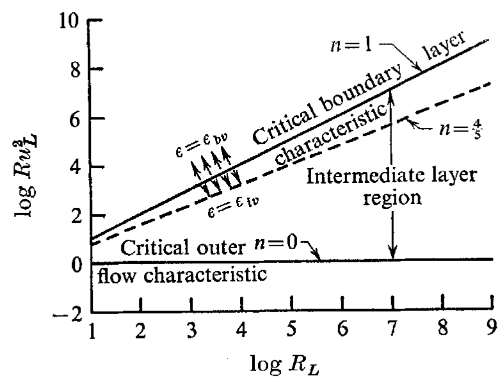

FIGURE 3. The various flow régimes in Russell number–Reynolds number parameter space.

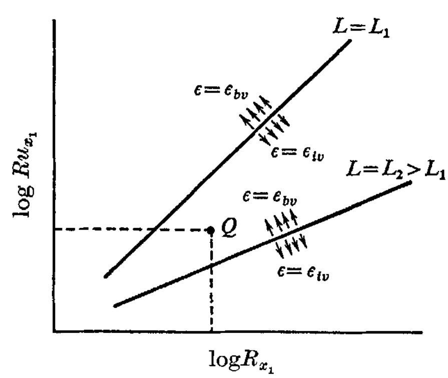

FIGURE 4. The qualitative effect of the plate length on the critical ($n = 1$) boundary-layer flow characteristic.

Another useful representation of the flow is obtained by writing the Russell–Reynolds number relation (28) in terms of the running length x_1

$$Ru_{x_1}^2 = \left(\frac{x_1}{L}\right)^{2-n} R_{x_1}^n = R_L^{n-2} R_{x_1}^2.$$

The magnitude of the Reynolds number based on the total plate length and the *relative* magnitude of the Russell and Reynolds numbers (characterized by n) define the slope of the flow characteristics in the two-dimensional parameter space $Ru_{x_1} - R_{x_1}$. Suppose we observe the flow at a fixed position on a plate, which we denote as the point Q in figure 4. Furthermore, suppose that this point lies below the critical boundary-layer characteristic ($n = 1$) for a plate of length L_1 so that the boundary layer grows in the downstream direction. Then, if the plate length is increased to L_2, $L_2 > L_1$, the slope of the critical characteristic decreases and we see that, if L_2 is sufficiently large, self-blocking occurs and an upstream wake and upstream growing boundary layer appear. Hence, for given flow conditions, one can always find a plate sufficiently long so that blocking occurs.

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